

Atom in a q -Analog Harmonic Oscillator Trap

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Abstract

We study the population inversion and Q -function of a two-level atom, interacting with single-mode laser light field, in a q -analog harmonic oscillator trap for increasing q . For $\tau = .003(q = e^\tau)$ the collapses and revivals of popu-

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lation inversion become well defined facilitating experimental observation but for large $\tau \sim 0.1$ the time dependence of population inversion is completely wiped out.

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I. INTRODUCTION

Recent development of quantum groups [1–3] has resulted in the construction of q -analogs of quite a few quantum systems including a very popular system that is the harmonic oscillator. A q -analog realization of harmonic oscillator has been given by Macfarlane [4] and Biedenharn [5]. In Ref. [5] coherent analogs of Glauber states have also been proposed. It is of great interest to use these quantum analog formulations to study physical systems and there are situations where a description in terms of q - analog structures is apparently more natural. The case of an atom being cooled through interaction with the radiation offers an interesting case study. It has been shown [6] recently that the atom can be cooled down to energies very close to its zero point vibrational energy. At these energies the quantum effects due to center of mass motion of the atom are expected to be very important [7]. Blockley et. al. [8] have studied the collapses and revivals of population inversion in a single two-level atom interacting with a classical single mode travelling light field while constrained to move in a one-dimensional harmonic Oscillator trap. They also discuss the possibility of observing the collapses and revivals experimentally.

The interaction of the atom with the trap potential results in energy exchange between the internal degrees of freedom of the atom and the center of mass motion. As the atom slows down, the amount of energy exchanged in each step of the cooling process is not expected to be constant but a variable dependent on the initial energy state of the atomic center of mass. Presently we study the time evolution of a two-level atom in a q -deformed harmonic oscillator trap, in interaction with a single mode travelling light field. In a q -deformed harmonic oscillator trap, the energy spacing between the trap states, as seen by the atom, is a function of the initial vibrational state of the atom. As such, when there is a lot of energy associated with center of mass motion energy loss and gain occurs in large energy quanta. However as the atom cools down, the energy exchange takes place in smaller units of energy. Besides that the relative energy separation between successive states is determined by the value of the deformation parameter. Blockley et al [8] have shown that in

the Lamb-Dicke regime, when only the interaction between nearest neighbours is significant the model is similar to the Jaynes-Cummings Model(JCM) [9] with trap quanta playing a role similar to that of the light quanta in JCM. An atom with its center of mass in a coherent trap state initially, shows collapses and revivals of its population inversion. Our object is to investigate the system response for a trap potential expected to be closer to experimental situation in comparoson with a harmonic oscillator trap. The energy spacings between energy eigen states of the q -analog harmonic oscillator are not constant but are a function of q -deformation. As such for a given system a suitable choice of the deformation parameter can take us from the classical limit where the trap states are closely spaced to the Lamb- Dicke regime with well spaced trap states. In our earlier attempts [10,11] to understand the physical nature of deformation in the context of pairing of nucleons, it has been found that the deformation amounts to simulating the nonlinearities of the problem or part of the interaction not included in the hamiltionian. With these results in mind we expect the -atom in q deformed oscillator trap model- to be a better description of the physical situation involved.

II. q -ANALOG HARMONIC OSCILLATOR TRAP

Consider a single two level atom having atomic transition frequency ω_a in a quantized q -analog quantum harmonic oscillator trap(q -deformed harmonic oscillator trap) interacting with a single mode travelling light field. The creation and annihilation operators for the trap quanta satisfy the following quocommutation relations,

$$aa^+ - qa^+a = q^{-N} ; \quad Na^+ - a^+N = a^+ ; \quad Na - aN = -a \quad (1)$$

Here N is the number operator. The operators a and a^+ act in a Hilbert space with basis vectors $|n\rangle$, $n = 0, 1, 2, \dots$, given by,

$$|n\rangle = \frac{(a^+)^n}{([n]_q!)^{\frac{1}{2}}} |0\rangle \quad (2)$$

such that $N|n\rangle = n|n\rangle$. The vacuum state is $a|0\rangle = 0$. We define here $[x]_q$ as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \quad (3)$$

and the q -analog factorial $[n]_q!$ is recursively defined by

$[0]_q! = [1]_q! = 1$ and $[n]_q! = [n]_q[n-1]_q!$. It is easily verified that

$$a^+ |n\rangle = [n+1]_q^{\frac{1}{2}} |n+1\rangle ; a |n\rangle = [n]_q^{\frac{1}{2}} |n-1\rangle \quad (4)$$

and N is not equal to a^+a . Analogous to the harmonic oscillator one may define the q -momentum and q -position coordinate

$$P_q = i\sqrt{\frac{m\hbar\omega}{2}}(a^+ - a) ; Q_q = \sqrt{\frac{\hbar}{2m\omega}}(a^+ + a). \quad (5)$$

The q -analog harmonic oscillator hamiltonian is given by

$$H_{qho} = \frac{1}{2}\hbar\omega(aa^+ + a^+a) \quad (6)$$

with eigenvalues

$$E_n = \frac{1}{2}\hbar\omega([n+1]_q + [n]_q). \quad (7)$$

For $\tau \ll 1$ we have for the energy spacing

$$E_{n+1} - E_n \sim \hbar\omega + (n+1)^2\tau^2\hbar\omega. \quad (8)$$

As the atomic state changes by absorption or emission of a light quantum there is a change in its vibrational state in the trap as well. We note that the trap states are not evenly spaced, the energy spacing being a function of deformation. Besides that as we move up in the number of vibrational quanta in the states the spacing between successive states increases. For a given region of vibrational states a suitable choice of deformation parameter should correspond to the Lamb Dicke regime.

III. DYNAMICS

The Hamiltonian for the system that consists the atom vibrating in the trap and interacting with a classical single-mode light field of frequency ω_l . is given by [8]

$$H = \frac{1}{2}\hbar\omega(a^+a + aa^+) + \frac{1}{2}\hbar\Delta\sigma_z + \frac{1}{2}\hbar\Omega(F\sigma^+ + F^*\sigma^-) \quad (9)$$

where $\Delta = \omega_a - \omega_l$, is the detuning parameter and Ω is the Rabi frequency of the system. The operator F stands for $\exp(ikQ_q) = \exp[i\epsilon(a^+ + a)]$. The parameter $\epsilon = \sqrt{\frac{E_r}{E_t}}$ is a function of the ratio of the recoil energy of the atom $E_r = \frac{\hbar^2 k^2}{2m}$ and the characteristic trap quantum energy $E_t = \hbar\omega$ in the limit $q \rightarrow 1$. Here k is the wave vector of the light field. The second term in the Hamiltonian refers to the energy associated with internal degrees of freedom of the atom, whereas the third term is the interaction of the atom with the light field. For nonzero values of deformation parameter q , the kinetic energy and potential energy of the atom in the trap are a function of deformation. In addition the interaction of the atom with the light field is also deformation dependent.

The state of the system at a time t ,

$$\Psi(t) = \sum_m g_m(t) |g, m\rangle + \sum_m e_m(t) |e, m\rangle \quad (10)$$

is a solution of the time dependent Schrodinger equation

$$H\Psi(t) = i\hbar \frac{d}{dt} \Psi(t). \quad (11)$$

The vector, $|g, m\rangle$, corresponds to the atom being in its ground state with its center of mass in m -th trap state. The label e stands for the excited state of the atom.

The probability amplitudes g_m and e_m satisfy the following set of coupled equations

$$i\frac{dg_m}{dt} = \frac{1}{2}g_m(t) (\omega([m+1]_q + [m]_q) - \Delta) + \frac{1}{2}\Omega \sum_n e_n(t) \langle g, m | \sigma^- F^* | e, n \rangle \quad (12)$$

$$\frac{de_m}{dt} = \frac{1}{2}e_m(t) (\omega([m+1]_q + [m]_q) + \Delta) + \frac{1}{2}\Omega \sum_n g_n(t) \langle e, m | F \sigma^+ | g, n \rangle \quad (13)$$

In order to evaluate the matrix elements $\langle m | F^* | n \rangle = \langle n | F | m \rangle^*$, firstly we make use of a special case of Baker-Hausdorff Theorem to rewrite the operator F as a product of operators i.e we use the equality

$$\exp[i\epsilon(a^+ + a)] = e^{-(|\epsilon|^2[a^+, a])} e^{i\epsilon a^+} e^{i\epsilon a}.$$

In doing so we have substituted for the commutation relation of the operators a and a^+ the value of the commutator given in Eq.(1) in the limit $q \rightarrow 1$. This approximation considerably simplifies the further evaluation of matrix elements of F using the defining Eqs.(1,2) for the operators a , a^+ and the vectors $|n\rangle$. The final expression for $m \leq n$ is given as,

$$\langle m | F | n \rangle = \frac{e^{-|\epsilon|^2} (i\epsilon)^{n-m} [m]_{\frac{1}{q}}!}{[n]_{\frac{1}{q}}!} \sum_{k=0}^m \frac{(\epsilon)^{2k} (-1)^k [n]_q!}{k! (n-m+k)! [m-k]_q!} \quad (14)$$

Depending on the difference $n - m$ the matrix element $\langle m | F | n \rangle$ can be real, imaginary, positive or negative. For the special case of $n = m + 1$ and to first order in ϵ the matrix element reduces to $\langle m | F | n \rangle = i\epsilon \sqrt{[n]_q}$.

In the Lamb Dicke regime the model is analogous to q -analog of Jaynes-Cumming Model [9] with center of mass motion quanta playing a role similar to that of the quantized radiation field. As pointed out in ref. [8], the effective Rabi frequency for trapping model can be easily calculated in the lowest approximation. In a similar way, considering only those transitions involving single trap quantum exchange we can easily calculate the analog of the effective Rabi frequency in the deformed harmonic Oscillator trap. In case the driving field is tuned to the first vibrational sideband, $\Delta = \pm\omega$, all other transitions can be neglected if these are oscillating at sufficiently high frequencies. For $\omega \gg \Omega$, the rotating wave approximation can be used and the dynamical equations solved to give the eigenvalues and the effective Rabi frequency. In ref. [8] the effective Rabi frequency obtained is $\mu(m) = \sqrt{(\omega + \Delta)^2 + \Omega^2 \epsilon^2 (m+1)}$. For the q -deformed oscillator trap the effective frequency, to first order in ϵ , is a function of deformation given by $\mu(m) = \sqrt{[\frac{\omega}{2} [\cosh(2\tau(m+1)) + 1] + \Delta]^2 + \Omega^2 \epsilon^2 [m+1]_q}$.

IV. INITIAL CONDITIONS

The initial state center of mass motion of the atom is represented by q -analog Glauber Coherent state(GCS), while the atom is in the ground state. The q -analog of GCS is written as

$$|\alpha\rangle_q = \exp_q^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]_q!}} |n\rangle \quad (15)$$

with the q -exponential defined through

$$\exp_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}$$

The complex parameter α determines the average number of trap quanta associated with the state $|\alpha\rangle_q$.

V. POPULATION INVERSION AND QUASI-PROBABILITIES

In figures (1) and (2), we plot the population inversion as a function of redefined time parameter $t(= \frac{\Omega t}{2\pi})$ for $q = e^\tau$ with τ taking real values 0.0, 0.002, 0.003, 0.004, 0.006, 0.01 and 0.1 respectively. Besides that we also use the parameter values $\bar{\epsilon} = \frac{\epsilon}{\Omega} = 50$ and $\bar{\Delta} = \frac{\Delta}{\Omega} = -50$. For $\tau = 0.0$ the results obtained by Blockley et. al. [8] are reproduced, when a harmonic oscillator trap is used and mean number of trap quanta is $\bar{m} = 16$. In the numerical calculation the maximum value of m in eq.(10) is restricted to $m = 32$. In fig.(1) for $\tau = 0.002$, the first revival is seen to peak earlier than the one for $\tau = 0.0$ and a well defined collapse appears between the first and the second revival. For a very small value of $\tau = 0.003$ the collapse and revival pattern is seen to become much more pronounced(well defined) as compared to the undeformed case. However as the deformation is increased to $\tau = 0.004$ and 0.006, the pattern again becomes diffuse after the first revival. The time interval after which the first revival occurs is seen to become shorter as τ is increased as seen in Fig.(2). Another interesting feature of the collapses and revivals is that as the

deformation increases the collapse period of the system is seen to exhibit increasing level of coherent population trapping. For a large deformation, that in the context of this system is something like $q \geq e^{0.01}$, the time dependence of population inversion is considerably washed away. For still larger values of τ for example $\tau = 0.1$ in fig.(2) no time dependence of the system is seen anymore. Similar results are obtained when imaginary values of the parameter τ are used. It is expected because for very small deformations the values of $[x]_q$ for $q = e^\tau$ with τ real are very close to the values of $[x]_q$ for τ imaginary so long as the modulus of τ is the same.

To obtain further insight into the energy exchange between the atomic center of mass motion and the internal degrees of freedom of the atom, we calculate the quasi-probability distributions in the $\alpha_r - \alpha_i$ plane. The Quasi-probability or Q-function is defined as $Q(\alpha) = (1/\pi) \langle \alpha | \rho_{red} | \alpha \rangle_q$, ρ_{red} being the density matrix reduced for degrees of freedom of center of mass motion. Figure (3) shows in the upper part the time evolution of Q-function for zero deformation at $t_0 = 0.0, t_1 = 30, t_2 = 130$ and $t_3 = 160$ respectively. Starting with the center of mass initial state as a coherent state $|\alpha_0\rangle$, the initially single peaked Q-function splits into two peaked function counterrotating in the complex plane. At $t_3 = 160$, when two peaks collide a revival of inversion oscillation occurs. A similar plot for $\tau = 0.003$ is shown for $t_0 = 0.0, t_1 = 30$ and $t_2 = 129.6$ respectively. An interesting feature of the Q-function for $\tau = 0.003$ is the presence of as many as five peaks at $t_1 = 30$. A sequence of Q-function distributions, for $t = 5, t = 8, t = 9, t = 10, t = 15$ and $t = 20$ shows the spreading out and breaking up of the quasi-probability into as many as eight distinct peaks (for $t = 15$). The revival of Rabi-oscillations occurs when all peaks collide together into a single peak at $t_2 = 129.6$. We also plot in Fig.(3) the Q-function for $\tau = 0.004$ at $t = 10$ again showing as many as seven peaks. It is apparent that when trap potential is q -deformed the quasi-probability distribution is no longer symmetrically distributed about $\alpha_i = 0.0$ line. This asymmetry is a manifestation of unequal energy spacings between different trap states.

VI. CONCLUSIONS

This calculation shows that in a q -deformed oscillator trap for a suitable deformation the collapse and revival patterns become well defined. As proposed by Blockley et al. [8] the theoretical predictions in this case can be verified experimentally. From our earlier work [10,11] with physical systems we may conjecture that the deformation corresponds to the presence of some nonlinearity in the trapping potential. In this case small scale nonlinearities are expected to play a beneficial role facilitating experimental observation. But large nonlinearities can wash out the collapse and revival pattern completely. This result is very important for constructing -an atom in a trap system- with a high probability for observing collapses and revivals of population inversion.

In the study of interaction of a three level atom with radiation in Ref [12] it is shown that by preparing the atom in a special way the time dependent collapses and revivals are either greatly diminished or vanish all together. The initial preparation forces the atomic population to remain coherently trapped in this configuration. Apparently q -deformed initial state is similar to a dressed atomic states. We may conclude therefore that the nonlinearities of the trapping potential lead to effects similar to those obtained by setting initial atomic state amplitudes to selected values.

In Ref. [13] collapses and revivals have been studied for JCM with an intensity dependent coupling constant by using q -analog of harmonic oscillator to represent the radiation quanta. In their calculation the collapses and revivals are seen to become more diffuse as the deformation is increased. Our result is different from theirs in the sense that the collapses and revivals become well defined. The major difference in their calculation and ours is that they use the zero order hamiltonian to be a harmonic oscillator Hamiltonian whereas we have explicitly included the zero order spectrum of q -analog oscillator.

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FIGURES

FIG. 1. Population inversion versus $t(= \frac{\Omega t}{2\pi})$ for $\tau = 0.0, 0.002, 0.003$ and 0.004

FIG. 2. Same as in Fig.1 for $\tau = 0.006, 0.008, 0.01, 0.1$.

FIG. 3. Quasi-probability distribution in $\alpha_r - \alpha_i$ plane at $t_0 = 0.0, t_1 = 30, t_2 = 130, t_3 = 160$.
For $\tau = 0.003, t_2 = 129.6$ and for $\tau = 0.004$ all the peaks correspond to $t = 10$.

FIG. 4. Same as in FIG. 3 for $\tau = 0.003$ at times shown in the figure.

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